Dynamic Stability Criteria for Clamped Shallow Arches under Timewise Step Loads

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The nonlinear problem of dynamic instability of snap-through of clamped shallow arches subjected to timewise step loads is investigated. Established are separate sufficient conditions for stability and for instability. For arches with parameters in certain special ranges, these two sufficient conditions coincide and, thus, yield the necessary and sufficient condition for stability. Specific problems investigated are "simple" clamped arches under "simple" loads, clamped sinusoidal arches under uniformly distributed loads, clamped parabolic arches under uniformly distributed loads and clamped sinusoidal arches under concentrated loads which may be located eccentrically. The criteria given in this paper are reliable to the extent that no approximation is made or is required in the mathematical analysis.

Nomenclature

a_n	= certain coefficient associated with $Y_n(\xi)$
b	= dimensionless arch rise of the sinusoidal or the parabolic arch
b_1	= half of the dimensionless arch rise of a "simple" clamped arch
b_n	= nth mode component (dimensionless) in the undeformed shape
B	= dimensionless intensity of the uniform load
c	= dimensionless eccentricity of the concentrated load
c_{mn}	= coupling coefficient between $Y_m(\xi)$ and $Y_n(\xi)$
F	= dimensionless magnitude of the concentrated load
G, G_1	= dimensionless thrusts in the arches
$\bar{H}$	= dimensionless total energy
$P_{1,1}^{(1)}$, etc.	= various equilibrium configurations
$Q(\tau,\xi)$	= dimensionless transverse load
Q_1	= dimensionless amplitude of the "simple" load
Q_n	= nth mode component (dimensionless) of the load distribution
$u(\xi)$	= current shape (dimensionless) of the arch
$u_0(\xi)$	= undeformed shape (dimensionless) of the arch
$V(C)$	= a Liapunov function associated with a configuration C
$Y_n(\xi)$	= a base function
$\alpha_n(\tau)$	= coefficient of $Y_n(\xi)$ in the series expansion of deformation
$\bar{\alpha}_n$	= value of α_n associated with a specific equilibrium configuration
$\beta_n(\tau)$	= generalized velocity associated with $\alpha_n(\tau)$
ξ	= coordinate
$\xi_n(\tau)$	= perturbation of $\alpha_n(\tau)$ from $\bar{\alpha}_n$
$\Xi(\xi)$	= spatial distribution of the timewise step load
τ	= dimensionless time variable

1. Introduction

In a recent paper,[1] a general procedure is discussed for studying the snap-through type of instability of certain elastic, continuous, and autonomous systems. The procedure, while not a simple one to apply, has nevertheless been successfully carried out for some shallow arch problems in Refs. 2 and 3. In particular, the simply supported shallow arches subjected to timewise step loads have been studied in Ref. 3 with the results expressed in the form of sufficient conditions for stability against snap-through and sufficient conditions for instability of snap-through. In this sense, this paper may be regarded as an extension of Ref. 3. The motivation for investigating clamped arches is twofold. First, from a practical point of view, clamped end conditions are probably more meaningful; thus, it is desirable to obtain specific results for arches of this type and to compare both qualitatively and quantitatively these results against those for simply supported arches. Secondly, it is also desirable to know whether the method of nonlinear treatment used in Ref. 3 is applicable to clamped arches, or whether the method works only for simply supported arches because of the special simplicity of the boundary conditions involved. This question has been raised by some prominent workers in the field when the results of Ref. 3 were made known. In this paper, we answer this question in affirmative. As a matter of fact, the method can be adapted for arches with other types of boundary conditions as well.

With regard to the literature, without attempting to be exhaustive we cite here the papers by Fung and Kaplan,[4] Hoff and Bruce,[5] Gjelsvik and Bodner,[6] Schreyer and Masur,[7] Lock,[8] Humphreys,[9] Simitses,[10] Nachbar and Huang,[11] Huang and Nachbar,[12] and Hsu.[13] Since the contents of the most of these papers have been briefly referred to in Refs. 2 and 13, we shall dispense with any further discussions.

Assuming that the initial thrust is absent and that the supports at the arch ends are rigid, the equation of motion for a shallow arch may be written as[2]

$$\frac{\partial^2 u}{\partial \tau^2} = -\frac{\partial^4(u-u_0)}{\partial \xi^4} - \left\{\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\left[\left(\frac{\partial u_0}{\partial \xi}\right)^2 - \left(\frac{\partial u}{\partial \xi}\right)^2\right] d\xi\right\}\frac{\partial^2 u}{\partial \xi^2} - Q(\tau,\xi) \quad (1)$$

where all quantities are dimensionless; u_0 and u are, respectively, the undeformed and the current shapes of the arch measured from the ξ-axis, τ is the time variable, and $\xi = -\pi/2$ and $\xi = \pi/2$ are the locations of the two ends of the arch. Apart from a shift of the origin of the ξ-axis to the center of the span and different boundary conditions, the

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geometry of the arch may be seen in Fig. 1 of Ref. 2. The arch being clamped at the ends, the boundary conditions are $u = u_0$ and $\partial u/\partial\xi = \partial u_0/\partial\xi$ at $\xi = \pm\pi/2$. The arch is initially at rest. At $\tau = 0$, it is acted upon suddenly by a transverse load of some spatial distribution $\Xi(\xi)$ which is maintained constant thereafter. The load term $Q(\tau, \xi)$ in Eq. (1) can, therefore, be written as $H(\tau)\Xi(\xi)$ where $H(\tau)$ is the Heaviside step function. Our problem is to find under what conditions a snap-through will not occur and under what conditions a snap-through will definitely take place. The definition of snap-through will be given later.

When one deals with arches of arbitrary shape, the question of how to represent the displacement $(u - u_0)$ becomes an important one. For reasons which will become apparent later, the representation to be used in this paper is made in terms of a complete set of functions $Y_n(\xi)$, $n = 1, 2, \ldots$, which are defined in detail in Sec. 3 of Ref. 13. These functions are simply the static buckling eigenmodes of a simple column clamped at both ends, normalized in an appropriate fashion. Let us now express the deformed configuration $u(\tau, \xi)$ by

$$u(\tau, \xi) = u_0(\xi) + \sum_{m=1}^{\infty} \alpha_m(\tau)Y_m(\xi) \tag{2}$$

where α_m, $m = 1, 2, \ldots$, are, in general, functions of time. On substituting Eq. (2) into Eq. (1), multiplying the equation thus obtained by $2Y_n/\pi$ and integrating with respect to ξ from $-\pi/2$ to $\pi/2$, we get after rearrangement

$$\dot{\alpha}_n = \beta_n \qquad n = 1, 2, \ldots \tag{3}$$

$$\sum_{m=1}^{\infty} c_{mn}\dot{\beta}_m + \alpha_n - G(a_n\alpha_n + b_n) + Q_n = 0 \tag{4}$$

$$n = 1, 2, \ldots$$

where a dot above a symbol denotes a time derivative, β_n plays the role of a generalized velocity,

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Y'_nY'_n d\xi = -\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Y_nY''_n d\xi \tag{5}$$

$$b_n = -\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Y_n u''_0 d\xi \tag{6}$$

$$c_{mn} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Y_mY_n d\xi \tag{7}$$

$$G = -2\sum_{m=1}^{\infty} \alpha_m b_m - \sum_{m=1}^{\infty} a_m\alpha_m^2 \tag{8}$$

$$Q_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} Y_n\Xi d\xi \tag{9}$$

In Eqs. (5–9), primes denote differentiations with respect to ξ. We note here certain important features of the system of equations (3) and (4). Because of the fact that in general, $c_{mn} \neq 0$ for $m \neq n$, there is dynamic coupling between the modes. The modes are also coupled statically, but they are coupled statically only in a very special way, namely, through the factor G. One may describe these modes as quasi-decoupled statically. It will be seen later that this feature allows a great deal of simplification of the nonlinear analysis. If we look back to the case of simply supported arches,[3] we find that a Fourier representation of the displacement by a set of sine functions will not only quasi-decouple the modes statically but also decouple the modes dynamically. This favorable situation has led to a very simple and effective analysis of dynamic stability. For the clamped arches, this dynamic decoupling is no longer possible if Eq. (2) is used for the displacement representation. The question is whether this will cause complications in the stability analysis. Fortunately, the dynamic coupling between the modes does not present any difficulty in the analysis if the method given in Refs. 1–3 is followed. In applying that method, as long as the kinetic energy is known to be positive definite, it is immaterial whether the modes are coupled or uncoupled dynamically. As mentioned before, the representation of u is not unique. For example, instead of using $Y_n(\xi)$, one can use the modes of linear vibration of a simple beam clamped at both ends. If that is done, one finds that the modes are nicely decoupled dynamically but they are coupled statically in a very involved manner. A consequence of this is that the analysis of the equilibrium configurations becomes complicated and it is almost impossible to obtain a general qualitative study of arches of arbitrary shape. In this respect, a proper choice of the mode representation is, therefore, a crucial step.

To determine the stability and instability criteria of snap-through, we follow the procedure discussed in Refs. 1–3. It involves three steps. First, one finds the possible equilibrium configurations of the arch for $\tau > 0$ when the external load is acting on the arch. Next, the condition for local instability of the "reference" equilibrium configuration, from which a snap-through may occur, will be investigated. This investigation will yield a sufficient condition for instability of snap-through. Lastly, when the reference equilibrium configuration is locally stable and when there is at least one other locally stable equilibrium configuration, find the crucial neighboring equilibrium configuration which will control a sufficient condition for stability against snap-through. In what follows, we carry out such programs for a number of arches of specific shapes subjected to loads of certain specific spatial distributions.

2. "Simple" Clamped Arches under "Simple" Loads

We consider first a family of clamped arches with their u_0 given by

$$u_0 = b_1Y_1/a_1 = b_1(1 + \cos 2\xi),\ b_1 > 0 \tag{10}$$

and subjected to loads of spatial distribution

$$\Xi(\xi) = Q_1(\cos 2\xi)/a_1 = 4Q_1 \cos 2\xi \quad Q_1 \neq 0 \tag{11}$$

By Eqs. (6) and (9), we find that for arches and loads of this kind $b_1 \neq 0$, $b_n = 0$, $n = 2, 3, \ldots$, $Q_1 \neq 0$, $Q_n = 0$, $n = 2, 3, \ldots$. For the lack of better names, we call these arches "simple" clamped arches and the loads "simple" loads. The governing equations for the equilibrium configurations are obtained from Eq. (4). They are

$$\alpha_1 - G_1(a_1\alpha_1 + b_1) + Q_1 = 0 \tag{12}$$

$$\alpha_n(1 - G_1a_n) = 0 \quad n = 2, 3, \ldots \tag{13}$$

where

$$G_1 = -2b_1\alpha_1 - a_1\alpha_1^2 - \sum_{n=2}^{\infty} a_n\alpha_n^2 \tag{14}$$

The solutions of Eqs. (12) and (13) may be summarized as follows.

First Kind: There are equilibrium configurations with $\alpha_1 \neq 0$ but all other $\alpha_n = 0$. For equilibrium configurations of this kind G_1 and Q_1 are related by

$$a_1Q_1/b_1 = 1 \pm (a_1G_1 - 1)(1 - a_1G_1/b_1^2)^{1/2} \tag{15}$$

and the value of α_1 is given by

$$\alpha_1 = (Q_1 - b_1G_1)/(a_1G_1 - 1) \tag{16}$$

In Fig. 1, we have plotted a_1Q_1/b_1 vs G_1 for four different values of b_1, namely, $b_1 = 3^{1/2}/2$, $3/2$, $3^{1/2}$, $11/4$. From Fig. 1, it is seen that depending upon the values of b_1 and Q_1 there may be one or three equilibrium configurations of the present kind. There will be three if $(a_1Q_1 - b_1)^2/4 + (1 - b_1^2)^3/27 \leqq 0$.

For future reference, we shall label these equilibrium configurations as $P_{1,1}^{(1)}$, $P_{1,1}^{(2)}$, and $P_{1,1}^{(3)}$. Here, the subscript in front of the comma denotes the mode component present in the undeformed shape and the subscript behind the comma denotes the mode component present in the load distribution. The superscripts serve simply as labels. For $b_1 > 1$, $P_{1,1}^{(1)}$ lies on ABC segment of the curve in Fig. 1, $P_{1,1}^{(2)}$ on CDE and $P_{1,1}^{(3)}$ on EBF. We note here that as b_1 decreases towards 1 the loop becomes smaller. The loop vanishes at $b_1 = 1$ and does not exist for $b_1 < 1$. For $b_1 < 1$, we label those on AD as $P_{1,1}^{(1)}$ and those on DF as $P_{1,1}^{(3)}$. The right half of each loop in Fig. 1 is characterized by $G_1 > G_{1s}$ where $G_{1s} = (2b_1^2 + 1)/3a_1$. This means that for $P_{1,1}^{(2)}$, if exists, $G_1 > G_{1s}$. Evidently, for $P_{1,1}^{(1,3)}$, $G_1 < G_{1s}$.

Geometrically $P_{1,1}^{(1)}$ and $P_{1,1}^{(3)}$ represent two quite different types of configuration. To see this, let us first look at the values of α_1 associated with them. From Eqs. (10) and (2), it is obvious that if $\alpha_1 > -b_1/a_1$, then the deformation is not quite large enough to cancel the initial arching. On the other hand, if $\alpha_1 < -b_1/a_1$ then the arch configuration lies entirely on the other side of the axis joining the arch ends. By Eq. (16), $\alpha_1 \gtrless -b_1/a_1$ implies $a_1Q_1/b_1 \gtrless 1$ if $G_1 > 1/a_1 = 4$, and $a_1Q_1/b_1 \lessgtr 1$ if $G_1 < 1/a_1 = 4$. Thus, if we divide the plane of Fig. 1 into the conventional quadrants, but with point B as the origin, then in the first and third quadrants $\alpha_1 > -b_1/a_1$ while in the second and fourth quadrants $\alpha_1 < -b_1/a_1$. Noting now that $P_{1,1}^{(1)}$ is always in the first and third quadrants, one concludes that it represents an equilibrium configuration which still lies wholly on the original side. However, with $P_{1,1}^{(3)}$, it being always in the second and fourth quadrants, the arch configuration has gone over from its original position to the other side.

Second Kind: There are equilibrium configurations with $\alpha_1 \neq 0$, $\alpha_j \neq 0$ but all other $\alpha_n = 0$. They appear in pairs and will be labelled as $P_{1,1,j}^{(\pm)}$. Here the subscript behind the second comma denotes the additional mode present in the deformed configuration. The pair $P_{1,1,j}^{(\pm)}$ is characterized by $G_1 = 1/a_j$ and is located at

$$\alpha_1 = (a_jQ_1 - b_1)/(a_1 - a_j) \tag{17}$$

$$\alpha_j = \pm\{(b_1/a_1)^2 - (a_1Q_1 - b_1)^2a_j/a_1(a_1 - a_j)^2 - 1/a_j^2\}^{1/2} \tag{18}$$

The relation $G_1 = 1/a_j$ represents a straight vertical line in the $(a_1Q_1/b_1) - G_1$ plane. In Fig. 1, three such lines are shown for $j = 2$, 3, and 4. It is also evident from Eq. (18) that $P_{1,1,j}^{(\pm)}$ exist if and only if the expression inside the curved brackets is positive. For a given value of b_1, we may take into account of this restriction in using Fig. 1 by 1) not extending the straight lines at $G_1 = 1/a_j$ outside the loop associated with that b_1 value and 2) by not using those straight lines at all if they lie entirely outside the loop. Thus, for example, for $b_1 = 3^{1/2}$ the line segments at $G_1 = 1/a_3$ and $1/a_4$ are not applicable and the line segment at $G_1 = 1/a_2$ does not extend beyond the loop $BC_3D_3E_3B$.

The preceding two kinds exhausted all the possibilities. Consequently, one is able to locate all the equilibrium configurations for a given set of b_1 and Q_1.

Local Stability

Once an equilibrium configuration has been found, its local stability character may be investigated by the procedure described in Ref. 2. We construct a Liapunov function $V(C)$, [see Eqs. (20, 12, and 21) in Ref. 2],

$$V(C) = \bar{H}(C) - \bar{H}(P) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\beta_m\beta_n + J(C) + O(\xi_n^3) \tag{19}$$

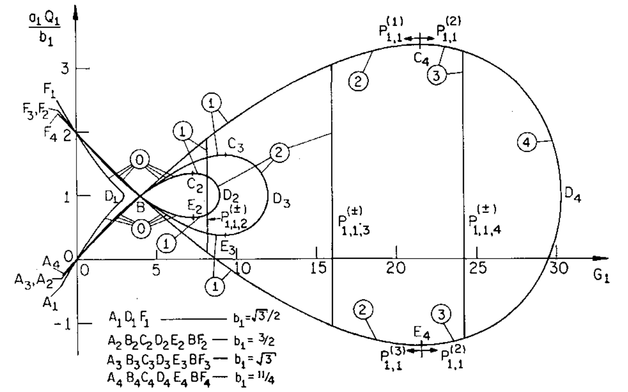

Fig. 1 The a_1Q_1/b_1 vs G_1 curves for locating all the equilibrium configurations of "simple" clamped arches under simple loads with $b_1 = 3^{1/2}/2$, $3/2$, $3^{1/2}$, and $11/4$.

where

$$\bar{H}(C) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\beta_m\beta_n + \sum_{n=1}^{\infty}\alpha_n^2 + 2\sum_{n=1}^{\infty}Q_n\alpha_n + \tfrac{1}{2}G_1^2 \tag{20}$$

is the total energy of the system, P denotes the equilibrium configuration under consideration and is assumed to be located at $\alpha_n = \bar{\alpha}_n$, $\beta_n = 0$, C is a neighboring configuration to P and has coordinates $\alpha_n = \bar{\alpha}_n + \xi_n$ and $\beta_n = \beta_n$, and

$$J(C) = \xi_1^2\left(1 + 2b_1^2 + 6b_1a_1\bar{\alpha}_1 + 3a_1^2\bar{\alpha}_1^2 + \sum_{m=2}^{\infty} a_1a_m\bar{\alpha}_m^2\right) + 4(b_1 + a_1\bar{\alpha}_1)\xi_1\sum_{m=2}^{\infty} a_m\bar{\alpha}_m\xi_m + \sum_{n=2}^{\infty}\xi_n^2\left(1 + 2a_nb_1\bar{\alpha}_1 + 2a_n^2\bar{\alpha}_n^2 + \sum_{m=1}^{\infty} a_na_m\bar{\alpha}_m^2\right) + \sum_{n=2}^{\infty}\sum_{\substack{m=2\\ m\neq n}}^{\infty} 2a_na_m\bar{\alpha}_n\bar{\alpha}_m\xi_n\xi_m \tag{21}$$

The quadratic form in β_n appearing in Eq. (20) is known to be positive definite. Therefore, in the neighborhood of P, the quadratic form $J(C)$ determines the type[1,2] of the equilibrium configuration P. The results are as follows: $P_{1,1,j}^{(\pm)}$ are always of type $(j - 1)$ and are, therefore, always locally unstable. $P_{1,1}^{(2)}$ is of type at least 1. The precise type may be determined by the following simple rule. If it has its associated G_1 value (see Fig. 1) in the range $1/a_n < G_1 < 1/a_{n+1}$, $n \geqq 2$, then it is of type n. In any case, $P_{1,1}^{(2)}$ is never locally stable. For $P_{1,1}^{(1)}$ and $P_{1,1}^{(3)}$, if the associated G_1 value is in the range $1/a_n < G_1 < 1/a_{n+1}$, $n \geqq 2$, then they are of type $(n - 1)$ and, hence, locally unstable. If the value of G_1 is less than $1/a_2$, then they are of type 0 and are, therefore, locally stable. We have shown in Fig. 1 by circled numbers the specific types of various possible equilibrium configurations for the cases of $b_1 = 3^{1/2}/2$, $3/2$, $3^{1/2}$, and $11/4$. The pattern is clear and interesting. Among the many features shown in Fig. 1, the most important one is the following. For a given value of b_1, as one moves to the right along the two branches passing through (0,0) and (0,2), respectively, the equilibrium configuration $P_{1,1}^{(1)}$ or $P_{1,1}^{(3)}$ is locally stable until it meets either the point of zero slope (point C or E) or the $P_{1,1,2}^{(\pm)}$ line segment, whichever is encountered first. Beyond that point $P_{1,1}^{(1,3)}$ are always locally unstable.

Snap-Through Instability

Next, we consider the main question of stability against snap-through. It is natural to take $P_{1,1}^{(1)}$ as the preferred configuration. In order to obtain a sharper picture of dynamic stability, we assume the existence of a small amount of positive damping so that the locally stable equilibrium configurations

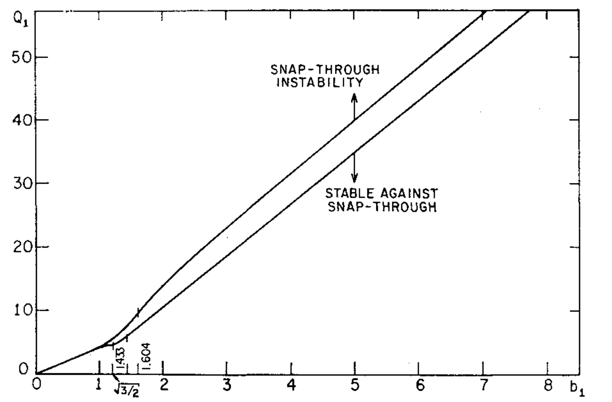

Fig. 2 Sufficient condition of instability and sufficient condition of stability against snap-through for a simple clamped arch under a simple load.

are asymptotically stable.§ By a snap-through instability, we have the following notion in mind. If after the step load is applied the system eventually settles down to $P_{1,1}^{(1)}$, then the system is said to be stable under that load against snap-through. If the system eventually takes on $P_{1,1}^{(3)}$, then the system is said to have snapped through.

From the study of local stability, we have found that for $b_1 > 1$ there is a load magnitude of Q_1 beyond which $P_{1,1}^{(1)}$ either does not exist or is locally unstable. In that case, with $P_{1,1}^{(3)}$ being the only locally stable equilibrium configuration the arch must eventually take on this snapped through configuration. This criterion gives us then a sufficient condition for instability of snap-through. The result is shown by the top curve in Fig. 2. For $b_1 < 1$, the existence of $P_{1,1}^{(1)}$ and the existence of $P_{1,1}^{(3)}$ are mutually exclusive; therefore, we can establish a necessary as well as sufficient condition for stability of snap-through. The result is also shown in Fig. 2. The top curve of Fig. 2 may also be given in the form of the following explicit formulas: For $0 < b_1 < 1$, the necessary and sufficient condition of stability against snap-through is

$$Q_1 < 4b_1 \tag{22}$$

For $1 \leqq b_1 < [(3a_1/a_2 - 1)/2]^{1/2} = 1.604$, a sufficient condition for instability is

$$Q_1 > 4[b_1 + 2(b_1^2 - 1)^{3/2}/(27)^{1/2}] \tag{23}$$

For $b_1 \geqq 1.604$, a sufficient condition for instability of snap-through is

$$Q_1 > 4[b_1 + (a_1/a_2 - 1)(b_1^2 - a_1/a_2)^{1/2}] \tag{24}$$

To find a sufficient condition for stability against snap-through for $b_1 > 1$, we follow the general procedure discussed in Refs. 1–3. To proceed we need the total energy expressions for the various equilibrium configurations. Here we list the following two:

$$\bar{H}(P_{1,1}^{(1)}) = -\frac{b_1^2}{a_1^2}\left[\left(1 - \frac{a_1G_1}{b_1^2}\right)^{1/2} - 1\right]^2 \times \left\{b_1^2 - \frac{3}{2}a_1G_1 + b_1^2\left(1 - \frac{a_1G_1}{b_1^2}\right)^{1/2} + 1\right\} \tag{25}$$

$$\bar{H}(P_{1,1,j}^{(\pm)}) = \frac{a_j}{a_1 - a_j}\left\{\frac{a_1Q_1 - b_1}{a_1} - \left(\frac{a_1}{a_j} - 1\right)\frac{b_1}{a_1}\right\}^2 - \frac{1}{2a_j^2} \tag{26}$$

§ A term representing the positive damping could be included in the equation of motion. However, this is not done here because the stability criteria to be established in this paper will not be affected by such a modification.

Omitting the detail of the analysis, we record below the sufficient conditions of stability against snap-through as well as the crucial equilibrium configurations P_* which control these criteria for various ranges of b_1. In the range $1 \leqq b_1 < (\frac{3}{2})^{1/2}$, P_* is $P_{1,1}^{(2)}$ and the sufficient condition against snap-through is

$$Q_1 < 4[b_1 - 2(b_1^2 - 1)(3b_1^2 - 3)^{1/2}/9] \tag{27}$$

In the range $(\frac{3}{2})^{1/2} \leqq b_1 < (3a_1/2a_2 - 1)(2a_1/a_2 - 2)^{-1/2} = 1.433$, P_* is again $P_{1,1}^{(2)}$ and the sufficient condition of stability against snap-through is

$$Q_1 < 4[2b_1(2b_1^2 + 9) + (2b_1^2 - 3)(4b_1^2 - 6)^{1/2}]/27 \tag{28}$$

In the range $(3a_1/2a_2 - 1)(2a_1/a_2 - 2)^{-1/2} \leqq b_1$, $P_{1,1,j}^{(\pm)}$ are P_* and the sufficient condition for stability against snap-through is

$$Q_1 < 4a_1\{b_1 - [(a_1/a_2 - 1)/2]^{1/2}\}/a_2 = 8.16(b_1 - 0.723) \tag{29}$$

These results of Eqs. (27, 28, and 29) are shown in Fig. 2 by the lower curve. The physical interpretation of Fig. 2 is as follows. Snap-through will definitely occur, if the load is above the upper curve and snap-through definitely will not occur for a load below the lower curve. In between these two curves, nothing definite can be said without further calculation. It is interesting to note that for $b_1 > 6.5$ the gap between the curves is less than 10% of the load. Moreover, the greater is the arch rise, the smaller is the gap, percentagewise, between these two curves.

In concluding this section, we wish to point out that from a practical point of view, the case of "simple" clamped arches under simple loads is not an important one because the load distribution is required to be of the shape of a complete cosine function over the span. However, the case does allow a complete, exact, and simple analysis; therefore, the clear picture of dynamic stability obtained for this case can serve as a guide for treating other more practical cases for which the results may not be so simple and direct. In the following sections, many of such cases will be investigated.

3. Clamped Sinusoidal Arches under Uniformly Distributed Loads

Next, let us consider clamped arches of sinusoidal shape subjected to timewise step loads which are uniformly distributed over the span. With the undeformed shape and the load expressed by

$$u_0(\xi) = b\cos\xi \tag{30}$$

$$\Xi(\xi) = B \tag{31}$$

where b and B are constants, the equation of equilibrium for $\tau > 0$, which is obtained from Eq. (1) by deleting the inertia term, may be written as

$$u'''' + Gu'' = u_0'''' - B \tag{32}$$

where

$$G = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}[(u'_0)^2 - (u')^2]d\xi \tag{33}$$

Although G depends upon the solution $u(\xi)$, it is a given constant, not a function of ξ, for a given solution $u(\xi)$. Thus, Eq. (32) may be integrated[7] and we obtain

$$u(\xi) = \frac{2bG^2 - B\pi(G-1)}{2G^{3/2}(G-1)\sin(\pi G^{1/2}/2)}\left(\cos G^{1/2}\xi - \cos\frac{\pi}{2}G^{1/2}\right) - \frac{b}{G-1}\cos\xi - \frac{B}{2G}\left(\xi^2 - \frac{\pi^2}{4}\right) \tag{34}$$

with G appearing as a parameter. To determine the parameter G, we substitute Eq. (34) into Eq. (33) and obtain

$$f\left(\frac{B}{b},G,b\right) \equiv \left(\frac{B}{b}\right)^2 \frac{1}{G^2}\left[\frac{\pi^2}{6} - \frac{4}{G} + \frac{\pi^2}{4\sin^2(\pi G^{1/2}/2)} + \frac{3\pi}{2G^{1/2}}\cot\left(\frac{\pi G^{1/2}}{2}\right)\right] - \frac{B}{b}\frac{1}{G-1}\left[\frac{2(G+1)}{(G-1)G^{1/2}}\cot\frac{\pi G^{1/2}}{2} + \frac{\pi}{\sin^2(\pi G^{1/2}/2)}\right] - \frac{G}{(G-1)^2}\left[G - 2 - \frac{G}{\sin^2(\pi G^{1/2}/2)} - \frac{2(G-5)G^{1/2}}{\pi(G-1)}\cot\frac{\pi G^{1/2}}{2}\right] + \frac{G}{b^2} = 0 \quad (35)$$

Equation (34) is the only solution provided that $G \neq 1/a_n = k_n^2$, $n = 2, 4, \ldots$. If $G = 1/a_j = k_j^2$ where j is an even integer, then, there will be an additional solution given by

$$u(\xi) = \frac{2b - \pi B a_j(1-a_j)}{2(1-a_j)k_j \sin(k_j\pi/2)}\left(\cos k_j\xi - \cos k_j\frac{\pi}{2}\right) - \frac{ba_j}{1-a_j}\cos\xi - \frac{Ba_j}{2}\left(\xi^2 - \frac{\pi^2}{4}\right) \pm k_j b\left[-f\left(\frac{B}{b},\frac{1}{a_j},b\right)\right]^{1/2} Y_j(\xi) \qquad j \text{ even} \quad (36)$$

Here, we note the appearance of $Y_j(\xi)$ in a very natural way. This is another indication that the functions $Y_n(\xi)$ provide us indeed with an appropriate set for deformation representation, if such a series representation is required.

Equation (34) yields equilibrium configurations which are symmetrical with respect to the center of the span, while the equilibrium configurations represented by Eq. (36) will contain antisymmetrical components. Equations (34) and (36) together exhaust all the possibilities of equilibrium configurations. These results may be shown graphically as in Fig. 3 where the looping curves represent Eq. (35) for different values of b and the vertical line segments located at $G = 1/a_j$ represent equilibrium configurations of Eq. (36) which exist only at these special values of G. For each given value of b, the line segments do not extend beyond the corresponding looping curve because otherwise the coefficient of $Y_j(\xi)$ in Eq. (36) becomes imaginary. For a given set of values of B and b, one first finds all the G values from Fig. 3 and then by Eqs. (34) and (36) determines the configurations. Thus, Fig. 3 serves the same purpose here as Fig. 1 does for simple clamped arches under simple loads.

In Fig. 3, the looping curves represent symmetrical equilibrium configurations. Let us now study the local dynamic stability of these configurations. It can be shown that all symmetrical equilibrium configurations with associated values of G greater than k_2^2 are locally unstable. The proof follows that used by Masur[14] in studying static buckling of shallow

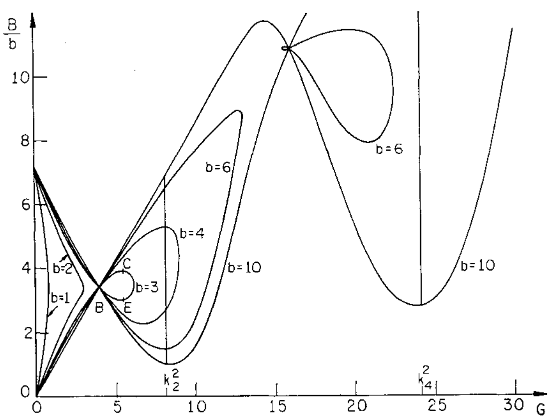

Fig. 3 The B/b vs G curves for locating all the equilibrium configurations of clamped sinusoidal arches under uniformly distributed loads with b = 1, 2, 3, 4, 6, and 10.

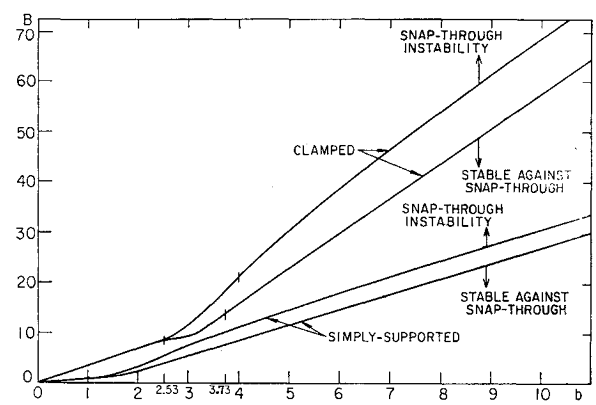

Fig. 4 Sufficient condition of instability and sufficient condition of stability for sinusoidal arches under uniformly distributed loads.

circular arches. On this point, see also Ref. 13. The proof is actually valid for all symmetrical deformations of symmetrical arches under symmetrical loads, not restricted to sinusoidal arches under uniform loads.

In Fig. 3, the origin represents the undeformed configuration of the arch under no load. As the load B is increased for a fixed b, the equilibrium configuration moves along the branch starting from 0 and going upward and to the right. It is natural to regard equilibrium configurations on that ascending branch as the preferred or reference configuration P_0, which will play the same role here as $P_{1,1}^{(1)}$ does for simple clamped arches under simple loads. There are two points on that branch which are important so far as the local stability is concerned. One is the point where $G = k_2^2$; beyond that point P_0 is locally unstable. The second point is the point C where $d(B/b)/dG = 0$, or the first point of local maximum on that branch. It can be shown that beyond the point C the equilibrium configuration P_0 is also locally unstable. Combining the foregoing two statements one finds that the sufficient condition for instability for P_0 is governed either by the point with $G = k_2^2$ or by the point C, whichever is encountered first on that ascending branch through 0. This result is shown by the top curve in Fig. 4. For small values of b (<2.53), the ascending branches in Fig. 3 do not extend beyond $G = 4$. For these values of b, the point with vertical slope is taken to corresponding to the necessary and sufficient condition of stability for P_0.

To find a sufficient condition of stability against snap-through, we follow the procedure described earlier. One finds that, for $b < 3.73$, the sufficient condition for stability is controlled by a locally unstable equilibrium configuration located on the right half of the first loop in Fig. 3, for instance, the part CE for $b = 3$. For $b > 3.73$, the sufficient condition is controlled by an equilibrium configuration located on the vertical line segment at $G = k_2^2$ in Fig. 3. The sufficient condition thus obtained for stability against snap-through is shown in Fig. 4 by the second curve from the top. To determine these sufficient conditions against snap-through, it is, of course, necessary to calculate the total energy associated with each equilibrium configuration. Closed form energy expressions have been derived; however, because of space limitation they will not be given here.

For comparison, we also have shown in Fig. 4 the sufficient condition for instability and the sufficient condition for stability for sinusoidal arches subjected to timewise step loads of uniform distribution but with simply supported ends. That set of curves is taken from Ref. 3. It is of interest to note that for a given b there are load magnitudes, represented by points in the region between the two middle curves, which will definitely cause snap-through for simply supported arches but are definitely not large enough to cause snap-through for clamped arches.

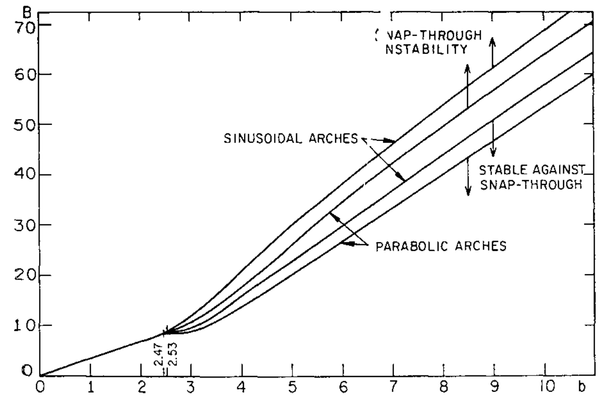

Fig. 5 Sufficient condition of instability and sufficient condition of stability for clamped sinusoidal arches and clamped parabolic arches under uniformly distributed loads.

4. Clamped Parabolic Arches under Uniformly Distributed Loads

For clamped parabolic arches subjected to timewise step loads of uniform spatial distribution, we take

$$u_0 = b(1 - 4\xi^2/\pi^2) \tag{37}$$

and the load as given by Eq. (31). Following the same procedure of analysis as that used in the last section, one obtains the sufficient conditions for stability and for instability as shown in Fig. 5 where the corresponding results for clamped sinusoidal arches are shown again for comparison. The abscissa in Fig. 5 is the arch rise at the center of the span, thus, the sinusoidal and the parabolic arches are compared for the same arch rise.

5. Clamped Sinusoidal Arches under Concentrated Loads

For the case of sinusoidal arches under concentrated loads that may be positioned eccentrically, we take the arch shape again as given by Eq. (30) and the load distribution as

$$\Xi(\xi) = F\delta(\xi - c) \tag{38}$$

where F is the load magnitude, $\delta(\xi)$ the Dirac delta function, and c the load eccentricity.

Again, the equations governing the possible equilibrium configurations and the total energies associated with these configurations can be found exactly and expressed in closed forms. Following the same general procedure used before, the stability and instability criteria of snap-through can be established. We present some of the results here in Figs. 6, 7, and 8. In Fig. 6, the criteria are shown for a range of b from 0 to 11 and for concentrated loads located at the center of the span. The ordinate used here is F/π, the average load intensity. This is done in order to make a comparison of these results against those for arches under uniformly distributed loads, which are also shown in Fig. 6. It is interesting to note that, for a given arch, there is a wide range of loads (total) such that if these loads are applied as concentrated ones then snap-through will definitely occur, while if they are applied as uniformly distributed ones snap-through will definitely not take place.

In Fig. 7, the curves which determine the equilibrium configurations are shown for cases with different amounts of load eccentricity. These curves are for $b = 8$. The pattern is clear. When $c \neq 0$ the load is no longer symmetric and, hence, there disappears the clean-cut bifurcations of equilibrium configurations at $G = k_2^2$. In Fig. 8, the sufficient conditions for stability and for instability for the case $b = 8$ are plotted against the load eccentricity. Two features may be noted here. First, the two curves merge into a single one at $c = 0.625\ (\pi/2)$. Thus, for $c > 0.625(\pi/2)$, we have the necessary and sufficient condition for snap-through stability. The reason for this is the disappearance of the first loop in the (F/b)-G curve in Fig. 7 or $c > 0.625(\pi/2)$. Consequently, the argument used in establishing Eq. (22) applies here. Second, we note that the curve of sufficient condition for stability against snap-through has a break in slope at $c = 0.384(\pi/2)$. This comes about because of the following reason: in the range $c < 0.384(\pi/2)$, the sufficient condition is controlled by the load at which $\bar{H}(P_*)$ changes from positive to negative as the load is increased, whereas for $0.384(\pi/2) < c < 0.625(\pi/2)$ the sufficient condition for stability is controlled by the first appearance of P_*, because $\bar{H}(P_*)$ is negative as soon as P_* appears. In this respect, the situation is very similar to that leading to the criterion of Eq. (27) for simple clamped arches under simple loads.

For the sake of comparison, we also have shown in Fig. 8 the corresponding curves for arches also of sinusoidal shape and also under concentrated loads but with simply supported ends. We note again that, for a given b, there are loads which will definitely cause snap-through if the arch is simply supported but definitely will not cause snap-through if clamped. Also, we note that in contrast to the clamped case the two curves for simply supported arches do not merge into one in the range indicated.

6. Conclusions

It can be seen easily from the results given in this paper that the criteria established here are of practical value be-

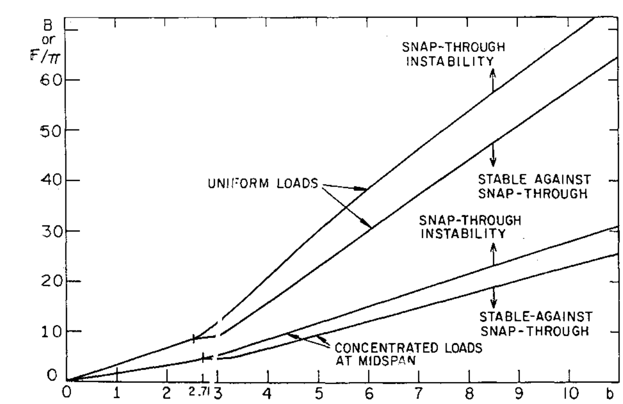

Fig. 6 Sufficient condition of instability and sufficient condition of stability for clamped sinusoidal arches under uniformly distributed loads and under concentrated loads at midspan.

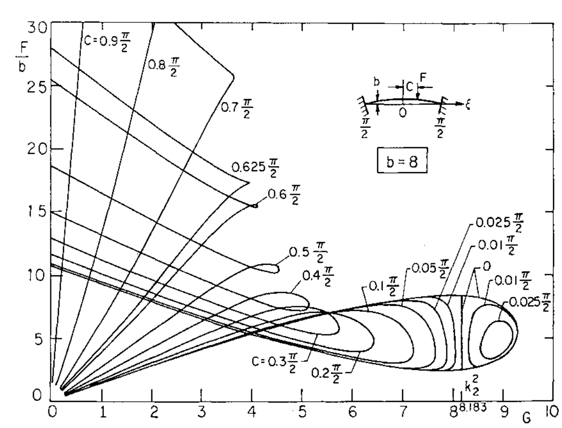

Fig. 7 The F/b vs G curves for locating all the equilibrium configurations of clamped sinusoidal arches under concentrated loads with different amounts of eccentricity and with $b = 8$.

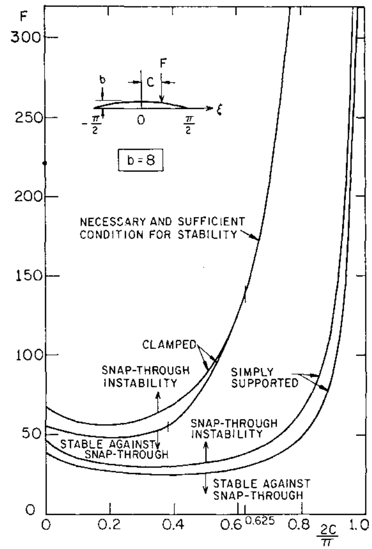

Fig. 8 Sufficient condition of instability and sufficient condition of stability for clamped or simply supported sinusoidal arches ($b = 8$) under concentrated loads.

cause, from a practical point of view, the zone of uncertainty is not large percentagewise. It is also evident that the method used in this paper for a number of specific arches can be just as effectively applied to other problems, such as parabolic arches under eccentrically placed concentrated loads, shallow arches of any shape under multiple concentrated loads, etc.

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